NEARLY EXOTIC TOPOLOGIES ON NORMED SPACES

BY
N. T. PECK

ABSTRACT

It is proved that for every infinite dimensional normed space $(E, \| \ \|)$ there is a non-trivial linear space topology τ on E which is weaker than the norm topology and is such that (E, τ) admits no non-trivial continuous linear functionals. If E is a space with a generalized basis or is a $C(X)$ space, it is proved that the topology τ can be taken to be Hausdorff.

In this note we investigate the existence of a non-trivial linear space topology τ on an infinite-dimensional vector space E such that (1) (E,τ) has no non-trivial continuous linear functionals, and (2) τ is weaker than some given norm topology on E. Following existing terminology (see [2]), we say that a linear space topology satisfying (1) is *nearly exotic*; two topologies τ_1 and τ_2 on a set are *orthogonal* if every non-empty τ_1-open set intersects every non-empty τ_2-open set.

In [2] Klee proved that if (E,τ) is an infinite-dimensional locally convex space and if τ is not a weak topology, then there is a non-trivial nearly exotic topology for E which fails to be orthogonal to τ. In [6] we conjectured that if $(E, \| \ \|)$ is an infinite-dimensional normed space, there is a non-trivial nearly exotic topology for E which is weaker than the norm topology. (Clearly such a topology will fail to be orthogonal to the norm topology.) We substantiate this conjecture and then turn to the more interesting question of the existence of a *Hausdorff* nearly exotic topology weaker than a given norm topology on an infinite-dimensional linear space. We show that every infinite-dimensional normed space with a generalized basis admits a weaker Hausdorff nearly exotic topology. We also show that every infinite-dimensional $C(X)$ space admits a weaker Hausdorff nearly exotic topology.

Throughout, *conv* A will denote the convex hull of the set A. The l_1 sum of

Received June 22, 1971 and in revised form August 17, 1971

normed spaces $(E_\alpha, \| \ \|_\alpha)$ (used in Corollary 1) is the set $\{x \in \prod_\alpha E_\alpha : \Sigma_\alpha \| x_\alpha \|_\alpha < \infty\}$; the number in this definition is the norm on the l_1 sum.

We thank H. Porta for many valuable discussions, and we thank the referee for greatly simplifying the original proofs of Corollaries 1 and 2.

The first result strengthens Lemma 3.2 of [6]:

LEMMA. *Let (E,τ_1) and (F,τ_2) be topological linear spaces with τ_2 Hausdorff and nearly exotic, and let T be a continuous linear map of E onto F. Let τ be the family of all sets in E of the form $T^{-1}(U)$ with U in τ_2. Then τ is a nearly exotic topology for E which is weaker than τ_1; the τ-closure of* 0 is $T^{-1}(0)$; *(whence) if T is $1-1$, τ is Hausdorff.*

PROOF. All the above statements are easily checked. We verify only that τ is nearly exotic: if U and V are τ_2-neighborhoods of 0, let x be an arbitrary element of E. Since τ_2 is nearly exotic there are non-negative real numbers $\{\lambda_i\}_{i=1}^n$ with $\Sigma_{i=1}^n \lambda_i = 1$ and elements $\{u_i\}_{i=1}^n$ of U such that $T(x) = \Sigma_{i=1}^n \lambda_i u_i$. For each i choose z_i in $T^{-1}(u_i)$. If $z = \Sigma_{i=1}^n \lambda_i z_i$, then $x = z + x - z \in \operatorname{conv} T^{-1}(U) + T^{-1}(0) \subset \operatorname{conv} T^{-1}(U) + T^{-1}(V)$. Thus x is in the τ-closed convex hull of every τ-neighborhood of 0, so τ is nearly exotic.

A *biorthogonal family* for a topological linear space E is a family (x_α, f_α) with $x_\alpha \in E, f_\alpha \in E^*$, such that $f_\alpha(x_\beta)$ is the Kronecker delta $\delta_{\alpha\beta}$ for each α and β. Also, S will denote the space of (equivalence classes of) Lebesgue-measurable functions on $[0,1]$ with the topology of convergence in measure. A metric for this topology is defined by

$$\sigma(f,g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\, d\mu$$

where μ is Lebesgue measure. It is well known that S is a complete Hausdorff nearly exotic space.

Theorem 3.3 of [6] can be strengthened as follows:

THEOREM. (a) *Every infinite-dimensional normed space $(E, \| \ \|)$ admits a weaker non-trivial nearly exotic topology;*

(b) *If in addition there is a biorthogonal family $\{x_\alpha, f_\alpha\}$ for E such that the set $\{f_\alpha\}$ is total over E, then $(E, \| \ \|)$ admits a weaker Hausdorff nearly exotic topology.*

PROOF. (a) Proposition 2.2 of [1] applied to any linearly independent sequence in E gives a biorthogonal sequence $(w_i, f_i)_{i=1}^\infty$ for E. Let the elements 1, sin nx,

$n = 1, 2, \cdots \cos nx$, $n = 1, 2, \cdots$ of S be arranged in a sequence $\{h_i\}_{i=1}^{\infty}$. Define $T: E \to S$ as follows:

$$T(x) = \sum_{i=1}^{\infty} 2^{-i} \frac{f_i(x)}{\|f_i\|} h_i.$$

For $p \leqq q$ we have the estimate

$$\sigma\left(\sum_{i=p}^{q} 2^{-i} \frac{f_i(x)}{\|f_i\|} h_i, 0\right) \leqq \sum_{i=p}^{q} \sigma\left(2^{-i} \frac{f_i(x)}{\|f_i\|} h_i, 0\right)$$

$$\leqq \sum_{i=p}^{q} \sigma(2^{-i}\|x\|, 0)$$

$$\leqq \left(\sum_{i=p}^{q} 2^{-i}\right)\|x\|, \text{ since}$$

the functions h_i are bounded by 1. This shows that the series defining T converges to an element of S and that T is continuous. Since $(w_i, f_i)_{i=1}^{\infty}$ is biorthogonal, the range of T contains the linear span of the sequence $\{h_i\}_{i=1}^{\infty}$, which is dense in S; hence the range of T is nearly exotic by Theorem 2.4 of [2]. Since S is Hausdorff, the Lemma gives the desired conclusion.

(b) Let $\{(x_\alpha, f_\alpha): \alpha \in A\}$ be a biorthogonal family for E so that the set $\{f_\alpha\}$ is total over E. Let $\{A_\gamma: \gamma \in \Gamma\}$ be a disjoint family of countably infinite sets whose union is A. Let $\{f_i^\gamma\}_{i=1}^{\infty}$ be an enumeration of the set $\{f_\alpha: \alpha \in A_\gamma\}$, and define $T: E \to \prod_{\gamma \in \Gamma} S$ by

$$(T(x))_\gamma = \sum_{i=1}^{\infty} 2^{-i} \frac{f_i^\gamma(x)}{\|f_i^\gamma\|} h_i,$$

where $\{h_i\}$ is defined as in (a) above. The argument in (a) shows that T is continuous. If H is the linear span of $\{h_i\}_{i=1}^{\infty}$ in S, the biorthogonality easily implies that the range of T contains the algebraic direct sum $\oplus \sum_{\gamma \in \Gamma} H$, which is dense in $\prod_{\gamma \in \Gamma} S$. Thus the range of T is nearly exotic since $\prod_{\gamma \in \Gamma} S$ is nearly exotic.

If $T(x) = 0$ then $(T(x))_\gamma = 0$ for all γ. The coefficients

$$\frac{2^{-i} f_i^\gamma(x)}{\|f_i^\gamma\|}$$

are easily seen to be the Fourier coefficients of $(T(x))_\gamma$; thus $f_i^\gamma(x) = 0$ for all γ and

i and hence by our assumption on $\{f_\alpha\}$, $x = 0$. Thus T is 1-1 and again the Lemma gives the conclusion.

A set $\{x_\alpha\}$ in a normed space E is a *generalized basis* for E if there is a corresponding set $\{f_\alpha\}$ in E^* so that $\{(x_\alpha, f_\alpha)\}$ is a biorthogonal family and $\{f_\alpha\}$ is total over E. If, in addition, the closed linear span of $\{x_\alpha\}$ is E, $\{x_\alpha\}$ is a *Markushevich basis* for E.

Every weakly compactly generated Banach space E has a Markushevich basis [3, p. 918]. Hence for such E, if E is infinite-dimensional both E and E^* have an infinite generalized basis and thus by the Theorem both have a weaker Hausdorff nearly exotic topology. Also, it is easy to see that if $\{E_\alpha\}$ is a family of Banach spaces each of which has a Markushevich basis, then the l_1 sum of the spaces E_α has a Markushevich basis.

COROLLARY 1. *Every infinite-dimensional AL-space admits a weaker Hausdorff nearly exotic topology.*

PROOF. Let E be an infinite-dimensional AL-space. By a theorem of Maharam [4] E can be written as the l_1 sum of spaces $L_1(v_\gamma)$, where each $L_1(v_\gamma)$ has one of these two forms: (i) v_γ is a strictly positive measure on the family of all subsets of a finite or countably infinite set; (ii) v_γ is a positive multiple of the product Lebesgue measure on $[0,1]^{m_\gamma}$, m_γ a non-zero cardinal number. Thus in either case $L_1(v_\gamma)$ is weakly compactly generated and so has a Markushevich basis. By the preceding remarks E has a Markushevich basis; an application of the Theorem completes the proof.

COROLLARY 2. *Every infinite-dimensional $C(X)$ space (X compact Hausdorff) admits a weaker Hausdorff nearly exotic topology.*

PROOF. Let $E = C(X)$. By the method of proof of Corollary 1, E^* has a Markushevich basis $\{x_\alpha, f_\alpha\}$; then $\{f_\alpha, x_\alpha\}$ is an infinite generalized basis for E^{**}. If this generalized basis is used in the theorem we obtain a 1-1 continuous inear map $T\colon E^{**} \to \prod_\gamma S_\gamma$, where each S_γ is S and T has a dense range. Since (in this case) T is defined by means of functionals in E^*, an examination of the construction in the Theorem shows that T is w^*-continuous on each norm-bounded subset of E^{**}. The unit ball of E is w^*-dense in the unit ball of E^{**}, so the restriction of T to E has a dense range in $\prod_\gamma S_\gamma$. By the theorem, the proof is complete.

REFERENCES

1. V. L. Klee, *On the borelian and projective types of linear subspaces*, Math. Scand. **6** (1958), 189–199.

2. V. L. Klee, *Exotic topologies for linear spaces*, Proc. Symposium on General Topology and its Relations to Modern Algebra, Prague, 1961, 238–249.

3. J. Lindenstrauss, *Decomposition of Banach spaces*, Indiana Univ. Math. Jour. **20** (1971), 917–919.

4. D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 108–111.

5. S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires*, Studia Math. **10** (1948), 184–208.

6. N. T. Peck, *On non locally convex spaces*, Math. Ann. **161** (1965), 102–115.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS